

# Davey-Stewartson description of two-dimensional nonlinear excitations in Bose-Einstein condensates

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We study nonlinear modulation of collective excitations in disk-shaped Bose-Einstein condensates with a repulsive interatomic interaction. Using a method of multiple scales we show that the nonlinear evolution of a wave packet, formed by the superposition of short-wavelength excitations, and a long-wavelength mean field, generated by the self-interaction of the wave packet, are governed by Davey-Stewartson (DS) equations. Consequently, two-dimensional soliton (dromion) solutions can develop and propagate. We further derive a set of DS equations with variable coefficients for the situation where a slowly varying trapping potential in transverse directions has been taken into consideration. Finally, the dromion solutions and their stability are investigated by numerical simulations.

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## I. INTRODUCTION

The remarkable experimental demonstration of Bose-Einstein condensation in weakly interacting atomic gases [1] has opened a new avenue for the investigation of nonlinear properties of matter waves. The most spectacular experimental progress achieved are the realization of matter-wave four-wave mixing [2], superradiance [3], and amplification [4], which have enabled the extension of linear atom optics to a nonlinear regime, i.e., nonlinear atom optics [5].

As an interesting nonlinear phenomenon, solitonlike nonlinear localized structures have received much attention in the past few decades. Solitons appear widely in solid, fluid and optical media and have potential applications in optical communications and information processing [6–9]. The matter-wave solitons, which have been observed recently in Bose-Einstein condensates [10–14], take also an important role in the study of Bose-Einstein condensation and matter waves [15–21].

Researches of soliton formations in Bose-Einstein condensates (BECs), so far have concentrated mainly on one-dimensional (1D) solitons moving in cigar-shaped traps. Obviously, to observe a two-dimensional (2D) soliton one will need a BEC formed in a disk-shaped trap which can be routinely produced in laboratory [22–24]. In a recent study, 2D weak nonlinear excitations under a long wavelength (or weak dispersion) approximation have been considered [25] and a Kadomtsev-Pitvashvili equation was derived to show that a lumplike 2D soliton excitation is possible. In addition to long wavelength excitations, however, BECs may also support nonlinear excitations with a short wavelength (i.e., with a strong dispersion) and other types of nonlinear excitations. This is the topic that will be addressed in the present work. We shall show that under suitable conditions a wave packet formed by superposition of short wavelength excitations can couple with a long-wavelength rectification field resulting from the self-interaction of the short wavelength excitations. We show that the nonlinear dynamics of the wave packet and the rectification field are controlled by non-

linearly coupled envelope equations, i.e., the Davey-Stewartson (DS) equations, which under suitable conditions allow dromionlike 2D soliton solutions.

The paper is organized as follows. In Sec. II we make an asymptotic expansion of the order parameter equation for a disk-shaped trapping potential and derive the DS equations using the method of multiple scales. In Sec. III analytical 2D soliton (dromion) solutions are provided and their properties are discussed. The numerical study on the dromion solutions and their stability in the presence of a slowly varying trapping potential in transverse directions are given in Sec. IV. The last section (Sec. V) contains the discussion and summary of our results.

## II. ASYMPTOTIC EXPANSION AND DS EQUATION

### A. The model

The dynamic behavior of a weakly interacting Bose gas at zero temperature is well described by the time-dependent Gross-Pitaevskii (GP) equation [1]

$$i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\Psi|^2 \right) \Psi, \quad (1)$$

where  $\Psi$  is an order parameter,  $\int d\mathbf{r} |\Psi|^2 = N$  is the number of atoms in the condensate,  $g = 4\pi\hbar^2 a_s / m$  is the interaction constant with  $a_s$  being the  $s$ -wave scattering length ( $a_s > 0$  for a repulsive interaction). We consider an anisotropic harmonic trap of the form

$$V_{\text{ext}}(\mathbf{r}) = \frac{m}{2} [\omega_{\perp}^2 (x^2 + y^2) + \omega_z^2 z^2], \quad (2)$$

where  $\omega_{\perp}$  and  $\omega_z$  are frequencies of the trap in the transverse ( $x$  and  $y$ ) and axial ( $z$ ) directions, respectively.

Expressing the order parameter in terms of its modulus and phase, i.e.,  $\Psi = \sqrt{n} \exp(i\phi)$ , we obtain a set of coupled equations for  $n$  and  $\phi$ . To make these equations be dimensionless, we let coordinates, time and density of condensate

be measured, respectively, by  $a_z, \omega_z^{-1}$ , and  $n_0$ , where  $a_z = [\hbar/(m\omega_z)]^{1/2}$  and  $n_0 = N/a_z^3$ . For a disk-shaped trap (i.e.,  $\omega_\perp \ll \omega_z$ ), one can assume [25,26]  $\sqrt{n} = P(x, y, t)G_0(z)$  and  $\phi = -\mu t + \varphi(x, y, t)$ , where  $G_0(z) = \exp(-z^2/2)$  is the dimensionless ground-state wave function of the 1D harmonic oscillator in the  $z$  direction,  $\mu$  is the dimensionless chemical potential, and  $\varphi$  is a phase function contributed from excitation, which is assumed to be a function of  $x$  and  $y$ . Then Eq. (1) is reduced to the following dimensionless (2+1)D equations of motion [25,26]:

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{\partial P}{\partial y} \frac{\partial \varphi}{\partial y} + \frac{P}{2} \left( \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) = 0, \quad (3)$$

$$-\frac{1}{2} \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) - \left( \mu - \frac{1}{2} \right) P + \left[ \frac{\partial \varphi}{\partial t} + V_\parallel(x, y) + \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial y} \right)^2 \right] P + Q' P^3 = 0, \quad (4)$$

where  $Q' = 2\sqrt{2}\pi N a_s / a_z$  is an effective interaction constant and  $V_\parallel(x, y) = (\omega_\perp / \omega_z)^2 (x^2 + y^2) / 2$  is the dimensionless trapping potential in the  $x$  and  $y$  directions.

In order to obtain useful insight of the possible 2D envelope soliton excitations we will neglect, as a first step, the slowly varying trapping potential [27]. The linear dispersion relation of the excitation can be obtained by assuming  $P = u_0 + a(x, y, t)$  ( $u_0 > 0$ ) with  $(a, \varphi) = (a_0, \varphi_0) \exp[i(k_1 x + k_2 y - \omega t)] + \text{c.c.}$  and with  $u_0, a_0$ , and  $\varphi_0$  being constants. It reads

$$\omega(k_1, k_2) = \frac{1}{2} k (4Q' u_0^2 + k^2)^{1/2}, \quad (5)$$

where  $k^2 = k_1^2 + k_2^2$ . Equation (5) is a Bogoliubov-type linear excitation spectrum in 2D. The dispersion of the system arises from quantum pressure [i.e.,  $k^2$  term in Eq. (5)]. The speed of sound is given by  $c = \lim_{k \rightarrow 0} [(\partial \omega / \partial k_1)^2 + (\partial \omega / \partial k_2)^2]^{1/2} = \sqrt{Q'} u_0$ , which is smaller than the speed of sound in a corresponding homogeneous system [i.e.,  $V_{\text{ext}}(\mathbf{r}) = 0$ ], where it takes the value  $\sqrt{2^{1/2} Q'} u_0$ .

### B. Asymptotic expansion and DS equations

We now investigate the weak, short wavelength nonlinear excitations based on the reduced equations (3) and (4). Following the line of Davey and Stewartson [28], we make the asymptotic expansion  $P = u_0 + \epsilon a^{(1)} + \epsilon^2 a^{(2)} + \epsilon^3 a^{(3)} + \dots$  and  $\varphi = \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + \epsilon^3 \varphi^{(3)} + \dots$ , and assume that  $a^{(j)}$  and  $\varphi^{(j)}$  ( $j = 1, 2, 3, \dots$ ) are the functions of the fast variable  $\theta = kx - \omega t$  and the multiple-scale slow variables  $\xi = \epsilon(c_g^{-1} x - t)$ ,  $\eta = \epsilon y$ , and  $\tau = \epsilon^2 t$ . Here,  $\epsilon$  is a small parameter characterizing the relative amplitude of the excitation and  $c_g$  is a constant yet to be determined. Substituting the above expansion into Eqs. (3) and (4), we obtain

$$-\omega \frac{\partial a^{(j)}}{\partial \theta} + \frac{1}{2} u_0 k^2 \frac{\partial^2 \varphi^{(j)}}{\partial \theta^2} = \alpha^{(j)}, \quad (6)$$

$$\left( -\frac{1}{2} k^2 \frac{\partial^2}{\partial \theta^2} + 2Q' u_0^2 \right) a^{(j)} - \omega u_0 \frac{\partial \varphi^{(j)}}{\partial \theta} = \beta^{(j)}, \quad (7)$$

for  $j = 1, 2, 3, \dots$ . The explicit expressions of  $\alpha^{(j)}$  and  $\beta^{(j)}$  are given in the Appendix (with  $\partial / \partial x_2 = \partial / \partial y_2 = 0$ ).

The lowest order ( $j = 1$ ) solution of Eqs. (6) and (7) can be obtained as

$$\varphi^{(1)} = A_0 + [A \exp(i\theta) + \text{c.c.}], \quad (8)$$

$$a^{(1)} = \frac{i u_0 k^2}{2 \omega} A \exp(i\theta) + \text{c.c.}, \quad (9)$$

where  $A_0$  describes a long-wavelength mean field (i.e.,  $k = 0$ ) required for cancelling a secular term appearing in high-order approximations, as will be seen below.  $A$  is an envelope function of the carrier wave  $\exp(i\theta)$  (with a short wavelength, i.e.,  $k \neq 0$ ). Both  $A_0$  and  $A$ , yet to be determined, are functions of the slow variables  $\xi, \eta$ , and  $\tau$ .  $\omega(k)$  is just the linear dispersion relation of the excitation, given in Eq. (5) with  $k_1 = k$  and  $k_2 = 0$ .

For the next order ( $j = 2$ ) solution, a solvability condition of Eqs. (6) and (7) requires that

$$c_g = \frac{d\omega}{dk} = \frac{1}{2\omega} (2Q' u_0^2 k + k^3), \quad (10)$$

i.e.,  $c_g$  is the group velocity of the carrier wave. The singularity-free second order solution is given by

$$\varphi^{(2)} = A_2 \exp(2i\theta) + \text{c.c.}, \quad (11)$$

$$a^{(2)} = B_{20} + [B_{21} \exp(i\theta) + B_{22} \exp(2i\theta) + \text{c.c.}], \quad (12)$$

with

$$A_2 = -i \frac{3 u_0 k^4 k^2 + 2Q' u_0^2}{2 \omega D(2k, 2\omega)} A^2, \quad (13)$$

$$B_{20} = \frac{1}{2Q' u_0} \frac{\partial A_0}{\partial \xi} - \frac{3 u_0 k^4}{4 \omega^2} |A|^2, \quad (14)$$

$$B_{21} = \frac{1}{2} Q' u_0^3 \frac{k^3}{\omega^3 c_g} \frac{1}{\partial \xi} \frac{\partial A}{\partial \xi}, \quad (15)$$

$$B_{22} = \frac{i \omega u_0}{k^2 + Q' u_0^2} A_2 + \frac{3k^4}{8\omega^2 k^2 + Q' u_0^2} A^2, \quad (16)$$

where  $D(k, \omega) = u_0 [-\omega^2 + \frac{1}{4} k^2 (k^2 + 4Q' u_0^2)]$ . From the expressions of (12) and (14) we see that the self-interaction of the short-wavelength excitations can result in a rectification, i.e., a mean field component (represented by the term proportional to  $|A|^2$ ) is generated. This type of rectification of the collective excitations is similar to the rectification in an optical medium [29].

For the  $j = 3$  order the solvability conditions of Eqs. (6) and (7) give rise to the following closed equations for  $A_0$  and  $A$ :

$$\alpha_1 \frac{\partial^2 A_0}{\partial \xi^2} - \frac{\partial^2 A_0}{\partial \eta^2} = \alpha_2 \frac{\partial}{\partial \xi} (|A|^2), \quad (17)$$

$$i \frac{\partial A}{\partial \tau} + \beta_1 \frac{\partial^2 A}{\partial \xi^2} + \beta_2 \frac{\partial^2 A}{\partial \eta^2} + \beta_3 |A|^2 A - \beta_4 A \frac{\partial A_0}{\partial \xi} = 0, \quad (18)$$

with coefficients

$$\alpha_1 = \frac{1}{c^2} - \frac{1}{c_g^2}, \quad (19)$$

$$\alpha_2 = \frac{k^2}{2c_p^3 c_g} (2c_p^2 + 3c_p c_g + c^2), \quad (20)$$

$$\beta_1 = \frac{1}{2\omega c_g^2} \left( c_p^2 - \frac{c^4}{c_p^2} + \frac{k^2}{4} \right), \quad (21)$$

$$\beta_2 = \frac{1}{2\omega} \left( c_p^2 + \frac{k^2}{4} \right), \quad (22)$$

$$\beta_3 = \frac{1}{2\omega} \left[ c^2 k^2 + \left( \frac{15}{8} \frac{c^2}{c_p^2} - \frac{1}{4} \right) k^4 + k^2 \left( 1 + \frac{3}{2} \frac{c^2}{c_p^2} \right) \times \frac{c_p^2 (2c^2 + k^2) + 3c^2 k^2 / 4}{c^2 + k^2} \right], \quad (23)$$

$$\beta_4 = \frac{1}{2\omega c_g} (c_g + 2c_p) k^2, \quad (24)$$

where  $c_p$  is the phase velocity defined by  $w(k)/k$ . We see that due to the nonlinear effect a coupling occurs between the envelope of the short wavelength excitations and the long wavelength rectification field. Equations (17) and (18) are the general form of the DS equations, which are reduced to a nonlinear Schrödinger equation if  $\partial/\partial\eta$  vanishes.

### III. DROMION SOLUTIONS

We now examine the 2D soliton solutions of the DS Eqs. (17) and (18). Using the transformation  $\partial A_0/\partial \xi = -\epsilon^{-2} \beta_1 k^4 / (\alpha_1 \beta_4) s$  and  $A = \epsilon^{-1} [4\beta_1 / (\alpha_2 \beta_4)]^{1/2} k^4 u$ , Eqs. (17) and (18) can be recast into

$$\frac{\partial^2 s}{\partial x'^2} - \frac{\partial^2 s}{\partial y'^2} + 4 \frac{\partial^2}{\partial x'^2} (|u|^2) = 0, \quad (25)$$

$$i \frac{\partial u}{\partial t'} + \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} + 2|u|^2 u + su = R[u], \quad (26)$$

where  $x' = (k^2/\sqrt{\alpha_1})(c_g^{-1}x - t)$ ,  $y' = k^2 y$ ,  $t' = (\beta_1 k^4/\alpha_1)t$ , and  $R[u] = (1 - \kappa_1) \partial^2 u / \partial y'^2 + 2(1 - \kappa_2) |u|^2 u$  with  $\kappa_1 = \alpha_1 \beta_2 / \beta_1$  and  $\kappa_2 = 2\alpha_1 \beta_3 / (\alpha_2 \beta_4)$ . For an arbitrary  $k$ , an exact 2D soliton solution of Eqs. (25) and (26) decaying in all spatial directions is not available yet. But we notice that for small  $k$  one has  $1 - \kappa_1 = -k^2/(6c^2) + O(k^4)$  and  $1 - \kappa_2 = -2k^2/(3c^2) + O(k^4)$ . Thus,  $R[u]$  is a quantity proportional to  $k^2$ . This implies that

$R[u]$  can be treated as a perturbation in the case of small  $k$ . As a first step, we assume  $k$  is small [30] and hence neglect  $R[u]$  and simplify Eqs. (25) and (26) to give

$$\frac{\partial^2 s}{\partial x'^2} - \frac{\partial^2 s}{\partial y'^2} + 4 \frac{\partial^2}{\partial x'^2} (|u|^2) = 0, \quad (27)$$

$$i \frac{\partial u}{\partial t'} + \frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial y'^2} + 2|u|^2 u + su = 0, \quad (28)$$

which are standard Davey-Stewartson-I (DSI) equations. The DSI equations (27) and (28) are completely integrable and can be solved exactly by the inverse scattering transform [31]. One of the remarkable properties of the DSI equations is that they allow explicit dromion solutions decaying in all spatial directions [31].

The dromion solution of the DSI equations (27) and (28) can be found as [32]

$$u = \frac{G}{F}, \quad s = 4 \frac{\partial^2}{\partial x'^2} \ln F, \quad (29)$$

where

$$F = 1 + \exp(\eta_1 + \eta_1^*) + \exp(\eta_2 + \eta_2^*) + \gamma \times \exp(\eta_1 + \eta_1^* + \eta_2 + \eta_2^*), \quad (30)$$

$$G = \rho \exp(\eta_1 + \eta_2), \quad (31)$$

with  $\eta_1 = (k_r + ik_i)(x' + y')/\sqrt{2} + (\Omega_r + i\Omega_i)t'$ ,  $\eta_2 = (l_r + il_i)(y' - x')/\sqrt{2} + (\omega_r + i\omega_i)t'$ ,  $\Omega_r = -2k_r k_i$ ,  $\omega_r = -2l_r l_i$ ,  $\Omega_i + \omega_i = k_r^2 + l_r^2 - k_i^2 - l_i^2$ ,  $\rho = |\rho| \exp(i\varphi_\rho)$ , and  $|\rho| = 2[2k_r l_r (\gamma - 1)]^{1/2}$ . Here,  $k_r, k_i, l_r, l_i, |\rho|, \varphi_\rho$ , and  $\gamma$  are real integration constants. If we choose  $k_r, l_r > 0$  we obtain  $\gamma = \exp(2\varphi_\gamma)$  with  $\varphi_\gamma > 0$ .

The explicit expression for the order parameter of the disk-shaped BEC in the case of the dromion excitation presented above is given by

$$\Psi = P(x, y, t) \exp \left[ -i\mu t - \frac{1}{2} z^2 + i\varphi(x, y, t) \right], \quad (32)$$

where

$$P(x, y, t) = u_0 \left( 1 - B_0 \frac{\sin \Phi}{n_1 \cosh f_1 + n_2 \cosh f_2} \right), \quad (33)$$

$$\varphi = - \frac{\beta_1}{\sqrt{\alpha_1 \beta_4}} k^2 D_0(x, y, t) + \left( \frac{4\beta_1}{\alpha_2 \beta_4} \right)^{1/2} k^4 \frac{4\sigma \cos \Phi}{n_1 \cosh f_1 + n_2 \cosh f_2}, \quad (34)$$

with

$$B_0 = 4\sigma \left( \frac{4\beta_1}{\alpha_2 \beta_4} \right)^{1/2} \frac{k^5}{\sqrt{k^2 + 4c^2}}, \quad (35)$$

$$\Phi = \left( k + (a-p) \frac{k^2}{\sqrt{\alpha_1}} \right) x + (a+p)k^2 y + \left( 2(\sigma^2 + \lambda^2 - a^2 - p^2) \right. \\ \left. \times \frac{\beta_1 k^4 - (a-p)c_g \frac{k^2}{\sqrt{\alpha_1}}}{\alpha_1} \right) t + \varphi_\rho, \quad (36)$$

$$f_1 = (\sigma + \lambda) \frac{k^2}{\sqrt{\alpha_1}} x + (\sigma - \lambda) k^2 y \\ - \left( (\sigma + \lambda) c_g \frac{k^2}{\sqrt{\alpha_1}} + 4(a\sigma - \lambda p) \frac{\beta_1 k^4}{\alpha_1} \right) t, \quad (37)$$

$$f_2 = (\sigma - \lambda) \frac{k^2}{\sqrt{\alpha_1}} x + (\sigma + \lambda) k^2 y \\ - \left( (\sigma - \lambda) c_g \frac{k^2}{\sqrt{\alpha_1}} + 4(a\sigma + \lambda p) \frac{\beta_1 k^4}{\alpha_1} \right) t + \varphi_\gamma, \quad (38)$$

where  $\sigma = k_r / \sqrt{2}$ ,  $\lambda = l_r / \sqrt{2}$ ,  $a = k_i / \sqrt{2}$ , and  $p = l_i / \sqrt{2}$ . The function  $D_0$  is given by

$$D_0 = \frac{\sigma \exp(\eta_1 + \eta_1^*) [1 + \gamma \exp(\eta_2 + \eta_2^*)] + \lambda \exp(\eta_2 + \eta_2^*) [1 + \gamma \exp(\eta_1 + \eta_1^*)]}{1 + \exp(\eta_1 + \eta_1^*) + \exp(\eta_2 + \eta_2^*) + \gamma \exp(\eta_1 + \eta_1^* + \eta_2 + \eta_2^*)} \quad (39)$$

with

$$\eta_1 + \eta_1^* = 2\sigma \left[ \frac{k^2}{\sqrt{\alpha_1}} x + k^2 y - \left( \frac{k^2}{\sqrt{\alpha_1}} c_g + 4a \frac{\beta_1 k^4}{\alpha_1} \right) t \right], \quad (40)$$

$$\eta_2 + \eta_2^* = 2\lambda \left[ -\frac{k^2}{\sqrt{\alpha_1}} x + k^2 y + \left( \frac{k^2}{\sqrt{\alpha_1}} c_g - 4p \frac{\beta_1 k^4}{\alpha_1} \right) t \right]. \quad (41)$$

From Eq. (33) we see that the excitation is a gray dromion created from the condensate background. The parameter  $B_0$  characterizes its grayness.

#### IV. NUMERICAL SIMULATIONS

In this section we study numerically dromion solutions in the presence of the trapping potential in transverse ( $x$  and  $y$ ) directions. We relax the constraint of small  $k$  (which is just for obtaining exact solutions presented in the last section) and hence give a general prediction of dromion excitations in the BEC with realistic physical conditions.

The time-dependent GP Eq. (1) is a (3+1)D system. For the disk-shaped harmonic trap (2) with  $\omega_\perp \ll \omega_z$ , one can reduce the dimension of the system by assuming that the order parameter takes the form  $\Psi = \psi(x, y, t) \exp(-z^2/2)$ , as it did in Sec. II A. Then, one obtains a (2+1)D, dimensionless equation of motion

$$i \frac{\partial \psi}{\partial t} = \left( -\frac{1}{2} \nabla_\perp^2 + \frac{1}{2} + V_\parallel(x, y) + Q' |\psi|^2 \right) \psi, \quad (42)$$

where  $V_\parallel(x, y)$  is the slowly varying trapping potential in the transverse directions and  $Q'$  is an effective interaction constant, given in Sec. II. Our numerical simulation is based on (2+1)D Eq. (42), which is equivalent to Eqs. (3) and (4). For exploring dromion solutions for nonvanishing  $V_\parallel(x, y)$  and

for larger  $k$ , in our simulation we take the dromion solution given by (32) as an initial condition. Our aim is to see how this dromion structure evolves with time.

The axial ground state wave function  $u_0$ , which has been taken as a constant for  $V_\parallel(x, y) = 0$  in the analytical approach given in the last two sections, is now a slowly varying function of  $x$  and  $y$ . Actually, it can be estimated by using the Thomas-Fermi approximation [25] and it takes the form

$$u_0(x, y) = \left\{ \frac{\sqrt{2} \omega_\perp}{Q \omega_z} \left[ \left( \frac{Q}{\sqrt{2} \pi^{3/2}} \right)^{1/2} - \frac{1}{2} \left( \frac{\omega_\perp}{\omega_z} \right) r^2 \right] \right\}^{1/2}, \quad (43)$$

where  $r^2 = x^2 + y^2$  and  $Q = 4\pi N a_s / a_z$ . The requirement that  $u_0$  is real gives the value of radius of the disk-shaped condensate, which reads  $l = \sqrt{\omega_z (2\mu - 1) / \omega_\perp}$ , with the dimensionless chemical potential  $\mu = 1/2 + \sqrt{Q' / \pi^{3/2}}$ . Shown in Fig. 1 is the profile of  $u_0$ , which is the background (ground state) condensate for nonvanishing  $V_\parallel(x, y)$ . In our computation with the parameters given below, we choose  $l = 80.57$ .

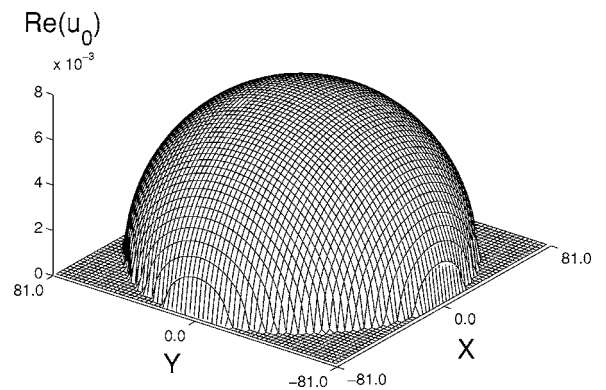


FIG. 1. The Thomas-Fermi profile of the ground-state wave function  $u_0(x, y)$ . The dimensionless radius  $l$  is 80.57.



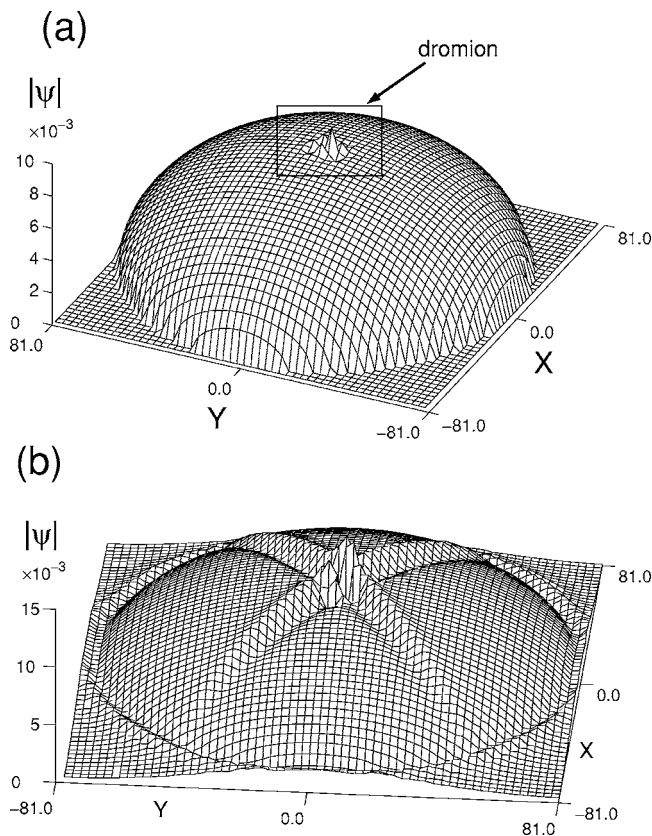


FIG. 2. (a) Dromion profile at  $t=0.40$  ms (initial time). Here, we plot just the shape of modulus  $\psi$  (namely,  $P$ ). The parameters are taken as  $\sigma=\lambda=a=p=1.0$ ,  $\varphi_p=0$  and  $\varphi_y=1.0$ , and  $k=0.5$ . (b) The dromion profile at  $t=1.40$  ms. Two dark solitons generate and the dromion rides on the cross-point of the dark solitons.

We carry out the computation in the region  $[-81.0, 81.0] \times [-81.0, 81.0]$  and the grid is taken as  $64 \times 64$  throughout the whole simulation. The space derivative in Eq. (42) is performed by using a pseudospectral method [33]. The time integration is performed by a fourth-order Runge-Kutta method for superior conservation of energy and other invariants [34,35]. The parameters of the system are taken as  $[16, 24]$   $\omega_{\perp}=2\pi \times 10$  Hz,  $\omega_z=2\pi \times 790$  Hz,  $a_s=2.7$  nm,  $a_z=0.74$   $\mu$ m, and  $N=2.9 \times 10^5$ . The initial parameters for the dromion are taken as  $\sigma=\lambda=a=p=1.0$ ,  $\varphi_p=0$ ,  $\varphi_y=1.0$ , and  $k=0.5$ . Figure 2 gives the profile of the dromion at  $t=0.40$  ms (initial time) in panel (a) and  $t=1.40$  ms in panel (b). We see that the dromion is fairly stable during propagation even for larger  $k$  and in the presence of the transverse trapping potential. However, there is a small deformation of the dromion in observable time interval 1.0 ms. The deformation is due to higher-order nonlinear and dispersion effects [see (11) and (12)], resulting in a generation of two dark solitons that are absent in the leading-order approximation. The dromion rides just on the cross-point of the dark solitons. In addition, a deformation of the background condensate is observed. Part of the atom penetrates the boundary of the trap potential [36], resulting in an attenuation of the background condensate. Figure 3 gives snapshots of dromion evolution from  $t=0.40$  ms to  $t=1.6$  ms with time interval 0.3 ms.

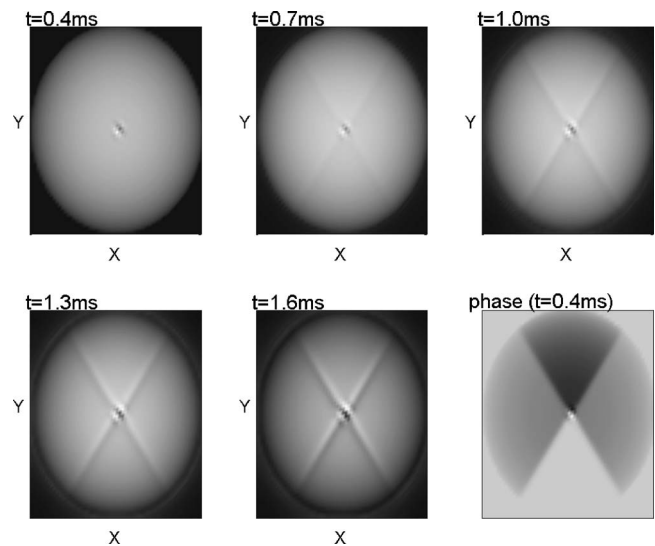


FIG. 3. Propagating dromion from  $t=0.40$  ms to  $t=1.60$  ms with time interval 0.3 ms. These sequential snapshots give the planforms of the modulus  $P$  with the same parameters as in Fig. 2. Shown in the last snapshot is the phase of order parameter  $\Psi$  for the dromion excitation at  $t=0.4$  ms.

## V. DISCUSSION AND SUMMARY

In the above derivation of the DS equations (17) and (18) we have neglected the slowly varying trapping potential in the transverse direction. This constraint can be relaxed further and the method of multiple scales still works. Of course, a new set of slow variables  $x_2=\epsilon^2x$  and  $y_2=\epsilon^2y$  are required to describe the variation of the trapping potential in the  $x$  and the  $y$  directions. In the Appendix we have provided a detailed derivation of modified DS equations with slowly varying coefficients and additional contributions.

In the present work we have shown that one can also obtain the DS equations and possible dromionlike excitations in a disk-shaped BEC. In addition to fluid dynamics [28,37], DS equations have also been derived in nonlinear optics [29], plasma physics [38], and lattice dynamics [39]. The DS equations are important not only because their solutions can provide many new 2D nonlinear localized excitations but also because their properties as a typical high-dimensional integrable system. These are very interesting aspects from the point of view of soliton theory [31].

To observe the dromion excitations in BECs described above, one should prepare a disk-shaped BEC with a repulsive interatomic interaction. The phase of the order parameter in the case of dromion is two cross kinks. One can use a 2D generation of the 1D phase imprinting technique used in producing 1D dark soliton in a cigar-shaped BEC [10,11] to generate a dromion.

In conclusion, we have investigated the nonlinear collective excitations in a disk-shaped BEC with a repulsive interatomic interaction. By using a method of multiple scales we have shown that the nonlinear evolution of a wave packet superposed by short-wavelength collective modes and a long-wavelength rectification field generated by self-interaction of the wave packet is governed by the DS equa-

tions, which allow 2D soliton (dromion) solutions. Variable-coefficient DS equations with additional terms are also derived when a slowly varying trapping potential in  $x$  and  $y$  directions is taken into account. The dromion solutions and their stability in the presence of slowly varying trapping potential in the transverse directions have also been investigated by numerical simulations.

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#### APPENDIX: DERIVATION OF THE MODIFIED DS EQUATIONS

In the presence of a trapping potential in the  $x$  and  $y$  directions, the asymptotic expansion presented in Sec. II B must be modified. For a disk-shaped trap we assume that the potential is a slowly varying function of  $x$  and  $y$ , i.e.,  $V_{\parallel}(x, y) = V_{\parallel}(x_2, y_2)$  with  $x_2 = \epsilon^2 x$  and  $y_2 = \epsilon^2 y$ . In the case of a harmonic potential, i.e.,  $V_{\parallel} = (\omega_{\perp} / \omega_z)^2 (x^2 + y^2) / 2$ , one has  $V_{\parallel} = \Omega_0^2 (x_2^2 + y_2^2) / 2$  under the assumption  $\omega_{\perp} / \omega_z = \epsilon^2 \Omega_0$  with  $\Omega_0$  a dimensionless constant of order unity. We make the following asymptotic expansion:

$$A = u_0 + \epsilon a^{(1)} + \epsilon^2 a^{(2)} + \epsilon^3 a^{(3)} + \dots, \quad (\text{A1})$$

$$\varphi = \epsilon \varphi^{(1)} + \epsilon^2 \varphi^{(2)} + \epsilon^3 \varphi^{(3)} + \dots, \quad (\text{A2})$$

where  $u_0 = u_0(x_2, y_2)$  and  $a^{(j)}$  and  $\varphi^{(j)}$  are the functions of the fast variable  $\theta = \int^x k(\epsilon^2 x', y_2) dx' - \omega t$  and the slow variables  $\xi = \epsilon [\int^x c_g^{-1}(\epsilon^2 x', y_2) dx' - t]$ ,  $\eta = \epsilon y$ ,  $\tau = \epsilon^2 t, x_2$ , and  $y_2$ , which are taken as independent of each other. Hence one has the following derivative expansions:

$$\frac{\partial}{\partial x} = k(x_2, y_2) \frac{\partial}{\partial \theta} + \epsilon c_g^{-1}(x_2, y_2) \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial x_2}, \quad (\text{A3})$$

$$\frac{\partial}{\partial y} = \epsilon \frac{\partial}{\partial \eta} + \epsilon^2 \left( \frac{\partial}{\partial y_2} + H(x_2, y_2) \frac{\partial}{\partial \theta} \right) + \epsilon^3 G(x_2, y_2) \frac{\partial}{\partial \xi}, \quad (\text{A4})$$

$$\frac{\partial}{\partial t} = -\omega \frac{\partial}{\partial \theta} - \epsilon \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau}, \quad (\text{A5})$$

where  $H(x_2, y_2) = (\partial / \partial y_2) \int^x k(\epsilon^2 x', y_2) dx'$ ,  $G(x_2, y_2) = (\partial / \partial y_2) \int^x c_g^{-1}(\epsilon^2 x', y_2) dx'$ .

Using (A1)–(A5), Eqs. (3) and (4) with  $V_{\parallel}(x, y) = V_{\parallel}(x_2, y_2)$  are transferred into

$$-\omega \frac{\partial a^{(j)}}{\partial \theta} + \frac{1}{2} u_0 k^2 \frac{\partial^2 \varphi^{(j)}}{\partial \theta^2} = \alpha^{(j)}, \quad (\text{A6})$$

$$\left( -\frac{1}{2} k^2 \frac{\partial^2}{\partial \theta^2} + 2Q' u_0^2 \right) a^{(j)} - \omega u_0 \frac{\partial \varphi^{(j)}}{\partial \theta} = \beta^{(j)}, \quad (\text{A7})$$

$$Q' u_0^2 = \mu - \frac{1}{2} - V_{\parallel}(x_2, y_2), \quad (\text{A8})$$

with

$$\alpha^{(0)} = 0, \quad (\text{A9})$$

$$\alpha^{(1)} = \frac{\partial a^{(1)}}{\partial \xi} - k^2 \frac{\partial a^{(1)}}{\partial \theta} \frac{\partial \varphi^{(1)}}{\partial \theta} - u_0 k c_g^{-1} \frac{\partial^2 \varphi^{(1)}}{\partial \theta \partial \xi} - \frac{1}{2} k^2 a^{(1)} \frac{\partial^2 \varphi^{(1)}}{\partial \theta^2}, \quad (\text{A10})$$

$$\begin{aligned} \alpha^{(3)} = & \frac{\partial a^{(2)}}{\partial \xi} - \frac{\partial a^{(1)}}{\partial \tau} - k \frac{\partial a^{(1)}}{\partial \theta} \left( k \frac{\partial \varphi^{(2)}}{\partial \theta} + c_g^{-1} \frac{\partial \varphi^{(1)}}{\partial \xi} \right) \\ & - k \frac{\partial \varphi^{(1)}}{\partial \xi} \left( k \frac{\partial a^{(2)}}{\partial \theta} + c_g^{-1} \frac{\partial a^{(1)}}{\partial \xi} + \frac{\partial u_0}{\partial x_2} \right) \\ & - \frac{1}{2} \left\{ u_0 \left[ 2k c_g^{-1} \frac{\partial \varphi^{(2)}}{\partial \theta \partial \xi} + \left( k \frac{\partial^2}{\partial \theta \partial x_2} + \frac{\partial}{\partial x_2} k \frac{\partial}{\partial \theta} \right. \right. \right. \\ & \left. \left. \left. + c_g^{-2} \frac{\partial^2}{\partial \xi^2} \right) \varphi^{(1)} \right] + a^{(1)} \left( k^2 \frac{\partial^2 \varphi^{(2)}}{\partial \theta^2} + 2k c_g^{-1} \frac{\partial^2 \varphi^{(1)}}{\partial \theta \partial \xi} \right) \right. \\ & \left. + k^2 a^{(2)} \frac{\partial^2 \varphi^{(1)}}{\partial \theta^2} \right\} - \frac{1}{2} u_0 \frac{\partial^2 \varphi^{(1)}}{\partial \eta^2}, \quad (\text{A11}) \end{aligned}$$

and

$$\beta^{(1)} = 0, \quad (\text{A12})$$

$$\begin{aligned} \beta^{(2)} = & k c_g^{-1} \frac{\partial^2 a^{(1)}}{\partial \theta \partial \xi} + u_0 \frac{\partial \varphi^{(1)}}{\partial \xi} + \omega a^{(1)} \frac{\partial \varphi^{(1)}}{\partial \theta} - \frac{1}{2} u_0 k^2 \left( \frac{\partial \varphi^{(1)}}{\partial \theta} \right)^2 \\ & - 3Q' u_0 (a^{(1)})^2, \quad (\text{A13}) \end{aligned}$$

$$\begin{aligned} \beta^{(3)} = & k c_g^{-1} \frac{\partial^2 a^{(2)}}{\partial \theta \partial \xi} + \frac{1}{2} \left( k \frac{\partial^2}{\partial x_2 \partial \theta} + \frac{\partial}{\partial x_2} k \frac{\partial}{\partial \theta} + c_g^{-2} \frac{\partial^2}{\partial \xi^2} \right) a^{(1)} \\ & + \frac{1}{2} \frac{\partial^2 a^{(1)}}{\partial \eta^2} + u_0 \left( \frac{\partial \varphi^{(2)}}{\partial \xi} - \frac{\partial \varphi^{(1)}}{\partial \tau} \right) - a^{(1)} \left( -\omega \frac{\partial \varphi^{(2)}}{\partial \theta} \right. \\ & \left. - \frac{\partial \varphi^{(1)}}{\partial \xi} \right) + \omega a^{(2)} \frac{\partial \varphi^{(1)}}{\partial \theta} - \frac{1}{2} \left[ 2u_0 k \frac{\partial \varphi^{(1)}}{\partial \theta} \left( k \frac{\partial \varphi^{(2)}}{\partial \theta} \right. \right. \\ & \left. \left. + c_g^{-1} \frac{\partial \varphi^{(1)}}{\partial \xi} \right) + k^2 a^{(1)} \left( \frac{\partial \varphi^{(1)}}{\partial \theta} \right)^2 \right] - Q' [6u_0 a^{(1)} a^{(2)} \\ & + (a^{(1)})^3]. \quad (\text{A14}) \end{aligned}$$

The explicit expressions of  $\alpha^{(j)}$  and  $\beta^{(j)}$  with  $j \geq 3$  are not needed and thus omitted here.

From Eq. (A8) one obtains  $u_0 = \{[\mu - 1/2 - V_{\parallel}(x_2, y_2)] / Q'\}^{1/2}$ . It is the ground state configuration of the condensate in the  $x$  and  $y$  directions. We see that for a slowly varying trapping potential the ground state has a shape of a Thomas-Fermi wave function. Equations (A6) and (A7) with  $j=1$  has the solution with the same form of (8) and (9), but the amplitude functions  $A_0$  and  $A$  are yet to be determined functions of  $\theta, \xi, \eta, \tau, x_2$ , and  $y_2$ . A solvability condition requires that

$$k = 2[(Q'^2 u_0^4 + \omega^2)^{1/2} - Q' u_0^2]^{1/2}. \quad (\text{A15})$$

Equation (A15) is the linear dispersion relation of the system, varying slowly with respect to  $x$  and  $y$ . Note that here for convenience we take the wave number  $k$  as the function of the frequency  $\omega$ .

In the next order ( $j=2$ ) Eqs. (A6) and (A7) give the solution being still with the form (11)–(16). The solvability condition in this order is still (10) but in the present case it should be understood as  $c_g = 1/(dk/d\omega)$ . The solvability condition in the order  $j=3$  yields the closed equations determining the amplitude functions  $A_0$  and  $A_1$ ,

$$\alpha_1 \frac{\partial^2 A_0}{\partial \xi^2} - \frac{\partial^2 A_0}{\partial \eta^2} = \alpha_2 \frac{\partial}{\partial \xi} (|A|^2), \quad (\text{A16})$$

$$i \frac{\partial A}{\partial \tau} + \beta_1 \frac{\partial^2 A}{\partial \xi^2} + \beta_2 \frac{\partial^2 A}{\partial \eta^2} + \beta_3 |A|^2 A - \beta_4 A \frac{\partial A_0}{\partial \xi} = -i \left( \beta_5 A + \beta_6 \frac{\partial A}{\partial x_2} \right). \quad (\text{A17})$$

The expressions of  $\alpha_1, \alpha_2, \beta_j$  ( $j=1, 2, 3, 4$ ) are still in the

form of those given in (19)–(24). The coefficients  $\beta_5$  and  $\beta_6$  take the form

$$\beta_5 = \frac{1}{8u_0 \omega k} \left( 4\omega^2 - 3k^4 + \frac{k^2}{k^2 + 2Q' u_0^2} (4\omega^2 + 5k^4) \right) \frac{\partial u_0}{\partial x_2}, \quad (\text{A18})$$

$$\beta_6 = \frac{4\omega^2 + k^4}{4\omega k^2}. \quad (\text{A19})$$

Equations (A16) and (A17) are variable-coefficient DS equations with additional terms, i.e., the terms on the right-hand side of Eq. (A17) contributed from the slowly varying trapping potential in the  $(x, y)$  plane. Similar DS equations (we call them the modified DS equations) have been obtained in weakly nonlinear water wave theory when considering a solitary wave propagating in a water channel with varying depth [40]. Obviously, if there is no trapping potential in the  $x$  and  $y$  directions, i.e.,  $V_{\parallel}=0$ , the modified DS equations (A16) and (A17) recover the form of (17) and (18).

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